Eur. Phys. J. B **11**, 309–315 (1999)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences
© Società Italiana di Fisica
Springer-Verlag 1999

Circular-like maps: sensitivity to the initial conditions, multifractality and nonextensivity

U. Tırnaklı[1,a], C. Tsallis[2], and M.L. Lyra[3]

[1] Department of Physics, Faculty of Science, Ege University, 35100 Izmir, Turkey
[2] Centro Brasileiro de Pesquisas Fisicas, Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro - RJ, Brazil
[3] Departamento de Fisica, Universidade Federal de Alagoas, 57072-970 Maceio-AL, Brazil

Received 5 February 1999

Abstract. Dissipative one-dimensional maps may exhibit special points (*e.g.*, chaos threshold) at which the Lyapunov exponent vanishes. Consistently, the sensitivity to the initial conditions has a power-law time dependence, instead of the usual exponential one. The associated exponent can be identified with $1/(1-q)$, where q characterizes the nonextensivity of a generalized entropic form currently used to extend standard, Boltzmann-Gibbs statistical mechanics in order to cover a variety of anomalous situations. It has been recently proposed (Lyra and Tsallis, Phys. Rev. Lett. **80**, 53 (1998)) for such maps the scaling law $1/(1-q) = 1/\alpha_{\min} - 1/\alpha_{\max}$, where $\alpha_{\min}$ and $\alpha_{\max}$ are the extreme values appearing in the multifractal $f(\alpha)$ function. We generalize herein the usual circular map by considering inflexions of arbitrary power z, and verify that the scaling law holds for a large range of z. Since, for this family of maps, the Hausdorff dimension d_f equals unity for all z in contrast with q which does depend on z, it becomes clear that d_f plays no major role in the sensitivity to the initial conditions.

PACS. 05.45.-a Nonlinear dynamics and nonlinear dynamical systems – 05.20.-y Statistical mechanics – 05.70.Ce Thermodynamic functions and equations of state

1 Introduction

Whenever a physical system has long-range interactions and/or long-range microscopic memory and/or evolves in a (multi)fractal-like space-time, the extensive, Boltzmann-Gibbs (BG) statistics might turn out to be inadequate in the sense that it fails to provide *finite* values for relevant thermodynamical quantities of the system. In order to theoretically deal with nonextensive systems of this (or analogous) kind, two major formalisms are available: the so-called quantum groups [1] and the generalized thermostatistics (GT), proposed by one of us a decade ago [2]. This two formalisms present in fact deep connections [3]. We focus here the GT. Within this framework, nonextensivity is defined through a generalized entropic form, namely

$$S_q = k\frac{1-\sum_{i=1}^{W} p_i^q}{q-1} \quad (q \in \mathcal{R}) \tag{1}$$

where k is a positive constant and $\{p_i\}$ is a set of probabilities associated to W microscopic configurations. We can immediately check that the $q \to 1$ limit recovers the usual, extensive, BG entropy $-k\sum_{i=1}^{W} p_i \ln p_i$. Also, if a composed system $A+B$ has probabilities which factorize into those corresponding to the subsystems A and B, then

$$S_q(A+B)/k = S_q(A)/k + S_q(B)/k + (1-q)S_q(A)S_q(B)/k^2.$$

This property exhibits the fact that q characterizes the degree of nonextensivity of the system.

Especially during the last five years, a wealth of works have appeared within this formalism. In fact, it is possible to (losely) classify these works as follows: (i) Some of them [4] address the generalization of relevant concepts and properties of standard thermostatistics, such as Boltzmann's H-theorem, fluctuation-dissipation theorem, Onsager reciprocity theorem, among others; (ii) Other GT works [5] focus applications to some physical systems where BG statistics is known to fail (stellar polytropes, turbulence in electron-plasma, solar neutrino problem, peculiar velocities of spiral galaxies, Levy anomalous diffusion, among others), and yields satisfactory results; (iii) Finally, an area of interest which is progressing rapidly, addresses the long standing puzzle of better understanding the physical meaning of the entropic index q. This line concerns the study of nonlinear dynamical systems (both low [6–9] and high [10,11] dimensional dissipative ones, as well as Hamiltonian systems [12]) in order to clarify the connection between q, the sensitivity to the initial conditions and a possible (multi)fractality hidden in the dynamics of the system. This paper belongs to the last class of efforts and is organized as follows. In Section 2

[a] e-mail: tirnakli@sci.ege.edu.tr

we briefly summarize recent related results. In Section 3 we introduce a new map which generalizes the circular one, and study its main properties. Finally, we conclude in Section 4.

2 Power-law sensitivity to initial conditions

The most important dynamical quantities that are used to characterize the chaotic systems are the Lyapunov exponent λ_1 and the Kolmogorov-Sinai entropy K_1 (the meaning of the subindex 1 will soon become transparent). Let us define, for a one-dimensional map of the real variable x, the quantity $\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)}$, where $\Delta x(0)$ and $\Delta x(t)$ are discrepancies of the initial conditions at times 0 and t respectively. It can be shown that, under quite generic conditions, ξ satisfies the differential equation $\mathrm{d}\xi/\mathrm{d}t = \lambda_1\,\xi$, hence $\xi(t) = \exp(\lambda_1 t)$. Consequently, if $\lambda_1 < 0$ ($\lambda_1 > 0$) the system is said to be *strongly* insensitive (sensitive) to the initial conditions. Similarly, for a dynamical system under certain conditions, we can define K_1 as essentially the increase, per unit time, of $S_1 \equiv -\sum_{i=1}^{W} p_i \,\ln\, p_i$. Furthermore, it can be shown that, with some restrictions, $K_1 = \lambda_1$ if $\lambda_1 \geq 0$ and $K_1 = 0$ otherwise. This is frequently referred to as the Pesin equality [14].

The case on which we are focusing in the present work is the so-called *marginal* case, corresponding to $\lambda_1 = 0$. It has been argued [6–8] that, in this case, the differential equation satisfied by ξ is $\mathrm{d}\xi/\mathrm{d}t = \lambda_q\,\xi^q$, hence

$$\xi(t) = [1 + (1-q)\lambda_q t]^{1/(1-q)}\,, \tag{2}$$

where λ_q is the generalized Lyapunov exponent. One can verify that $q = 1$ recovers the standard, *exponential* case whereas $q \neq 1$ yields a *power-law* behavior. If $q > 1$ ($q < 1$) the system is said to be *weakly* insensitive (sensitive) to the initial conditions. (Results which are consistent with the present $q < 1$ case have since long been observed [13] in some logistic-like maps). It is worth to mention that at the onset of chaos $\xi(t)$ presents strong fluctuations reflecting the fractal-like structure of the critical dynamical attractor and equation (2) delimits the power-law growth of the upper bounds of ξ.

For the marginal cases with $\lambda_1 = 0$, including period doubling, tangent bifurcation and chaos threshold, it has been introduced a generalized Kolmogorov-Sinai entropy K_q as the increase per unit time of S_q. It has been argued that there is a proper generalized entropy S_q with $q \neq 1$ that depicts a finite asymptotic variation rate at these marginal points . Finally, for this anomalous case, it has been proposed [6] that the Pesin equality itself can be generalized as follows: $K_q = \lambda_q$ if $\lambda_q \geq 0$ and $K_q = 0$ otherwise.

Recently, these ideas have been applied to some dissipative one-dimensional maps (a logistic-like and a periodic-like map, sharing the same universality class, as well as the standard circular map, which belongs to a different universality class), and the numerical results suggested a close relationship between the nonextensivity parameter q and the fractal (Hausdorff) dimension d_f associated with the dynamical attractor [7–9]. Very specifically, in those examples, when d_f approaches unity (which is the Euclidean dimension of the system) from below then q also approaches unity from below, and does that in a *monotonic manner*. Naturally, this fact strongly suggests that the validity of the statistical $q = 1$ (BG) picture is intimately related to the *full* occupancy of the phase space. However, an important question which remains open is whether the full occupancy is *sufficient* for having a BG scenario. We will show here that it is not!

Actually, the fractal dimension d_f gives a poor description of the complexity of critical dynamical attractors and a multifractal formalism is needed in order to reveal its complete scaling behavior. In this formalism each moment of the probability distribution (integrated measure) has a contribution coming from a particular subset of points of the attractor with fractal dimension f. Using a partition containing N boxes, the content on each contributing box scales as $N^{-\alpha}$. The multifractal measure is then characterized by the continuous function $f(\alpha)$ which reflects the fractal dimension of the subset with singularity strength α (for details see [15]).

Recently a new scaling relation has been proposed [8], namely

$$\frac{1}{1-q} = \frac{1}{\alpha_{\min}} - \frac{1}{\alpha_{\max}}, \tag{3}$$

where $\alpha_{\min}$ and $\alpha_{\max}$ are the extremes of the multifractal singularity spectrum $f(\alpha)$ of the attractor which govern respectively the scaling behavior of the more concentrated and more rarefied regions on the attractor. The above relation clearly indicates that once the scaling properties of the dynamical attractor are known, one can precisely infer the proper entropic index q that must be used for other purposes. In what follows we will introduce a new family of generalized circular maps characterized by an inflexion power z. For this family of maps, the critical attractors fully occupy the whole phase space, *i.e.*, $d_f(z) = 1$ for all z and nevertheless they exhibit a power-law sensitivity to initial conditions with $q < 1$. Further, we will compute the scaling properties of the critical attractors and show that our results are consistent with the scaling relation (3).

3 A family of circular-like maps

The circle map is an iterative mapping of one point on a circle to another of the same circle. This map describes dynamical systems possessing a natural frequency ω_1 which are driven by an external force of frequency ω_2; $\Omega \equiv \omega_1/\omega_2$ is known as the "bare" winding number. These systems tend to mode-lock at a frequency ω_1^* and $\omega \equiv \omega_1^*/\omega_2$ is known as the "dressed" winding number. The standard circle map, for one-dimension, is given by

$$\theta_{t+1} = \Omega + \theta_t - \frac{K}{2\pi}\sin(2\pi\theta_t) \quad \mathrm{mod}(1), \tag{4}$$

with $0 < \Omega < 1$; $0 < K < \infty$. For $K < 1$ the circle map is linear at the vicinity of its extremal point and

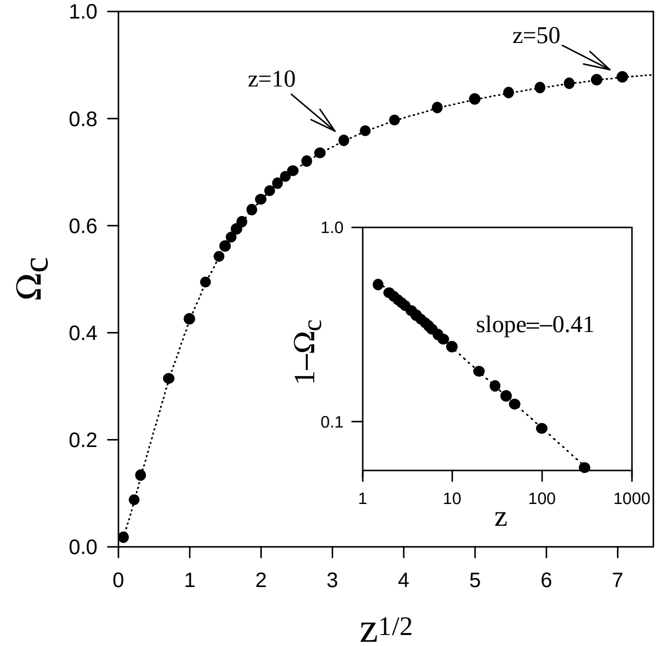


Fig. 1. The values of the "bare" winding number Ω_c as a function of z for the generalized circle maps.

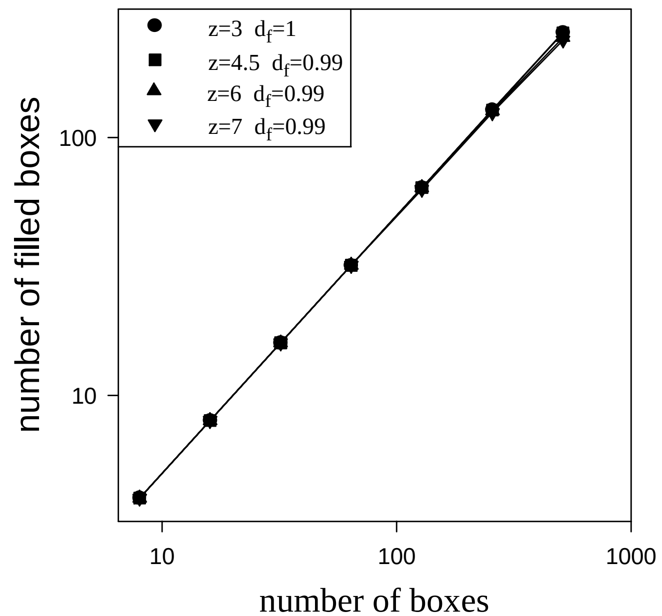


Fig. 2. Box counting graph for determining the fractal dimension d_f of the critical attractor for typical values of z.

exhibits only periodic motion. From now on we take $K = 1$, the onset value above which chaotic orbits exist. For this map, once mode-locked, $\omega = \lim_{t\to\infty}(\theta_{t+1} - \theta_t)$ remains constant and rational for a small range of the parameter Ω with the "dressed" *versus* "bare" winding number curve exhibiting a "devil staircase" aspect [16] (if $\theta_{t+1} < \theta_t$ then one shall use $\omega = 1+\theta_{t+1}-\theta_t$ in order to leave it mod(1)). At the onset to chaos, a set of zero measure and universal scaling dynamics is produced at irrational dressed winding numbers which have the form of an infinite continued-fraction expansion

$$\omega = \frac{1}{n + \frac{1}{m+\frac{1}{p+...}}}, \tag{5}$$

with $n, m, p, ...$ integers. The best studied one is when ω equals the golden mean, *i.e.*, $\omega_{\mathrm{GM}} = (\sqrt{5}-1)/2$ ($n = m = p = ... = 1$) [17,18]. It is worth mentioning that the golden mean is the asymptotic ratio between consecutive numbers of the Fibonacci series ($\lim_{n\to\infty} F_n/F_{n+1} = (\sqrt{5}-1)/2$, where $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$). In order to determine the bare winding number at this critical point, we iterate the map for a large number of time steps (10^6 steps) starting with $\theta_0 = 0$ and use a linear regression to numerically compute ω. The bare winding number Ω is then adjusted to have the renormalized winding number ω equal to the golden mean, which results in $\Omega_\mathrm{c} = 0.606661...$ With these parameters, the standard circle map has a cubic inflexion ($z = 3$) in the vicinity of the point $\bar{\theta} = 0$.

It is well known that the critical properties of circle maps depend on the order z of its inflexion point [19,20]. In this work we will investigate a generalized version of the circle map which can be defined as

$$\theta_{t+1} = \Omega + \left[\theta_t - \frac{1}{2\pi}\sin(2\pi\theta_t)\right]^{z/3}, \tag{6}$$

where $z > 0$ ($z = 3$ reproduces the standard case). For every value of z, the golden mean of ω corresponds to different "bare" winding numbers, which we call as Ω_c. In order to determine these critical values of Ω, one searches, within a given precision (12 digits in our calculations), the value of Ω corresponding to ω_{GM} with the same precision. The calculated values of Ω_c, as a function of z, are shown in Figure 1. The numerical values are indicated in Table 1. We remark that, in the limit $z \to 0$ ($z \to \infty$) we verify that $\Omega_\mathrm{c} \propto z^{1/2}$ ($1 - \Omega_\mathrm{c} \propto 1/z^\beta$ with $\beta \simeq 0.41$).

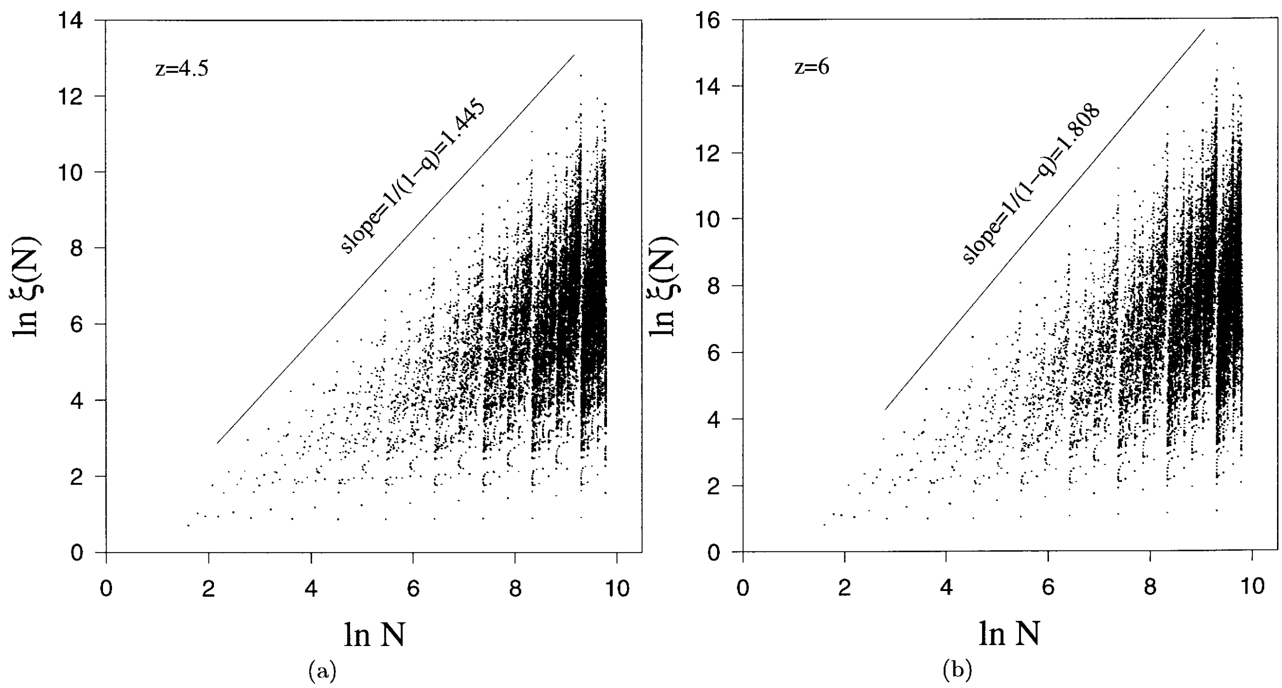
For our present purpose, a very important feature of this map is that, for every value of z, the critical attractor visits the *entire* circle ($0 < \theta_t \pmod 1 < 1$) and therefore has a support fractal (Hausdorff) dimension $d_f = 1$. The calculated values of d_f, using a box counting algorithm, are indicated in Figure 2 which shows that the Hausdorff and box counting dimensions of the critical attractor coincide. In what concerns the sensitivity to the initial conditions, the function $\xi(t)$ is given by

$$\ln\xi(t) = \ln\left|\frac{\mathrm{d}\theta_N}{\mathrm{d}\theta_0}\right| = \sum_{t=1}^{N}\ln\left\{\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\theta_t - \frac{1}{2\pi}\sin(2\pi\theta_t)\right]^{z/3}\right\}$$
$$= \sum_{t=1}^{N}\ln\left\{\frac{z}{3}\left[\theta_t - \frac{1}{2\pi}\sin(2\pi\theta_t)\right]^{\frac{z}{3}-1}[1-\cos(2\pi\theta_t)]\right\} \tag{7}$$

and displays, for $\Omega = \Omega_\mathrm{c}$, a power-law divergence, $\xi \propto t^{1/(1-q)}$, from where the value of q can be calculated by measuring, on a log-log plot, the upper bound slope,

Table 1. Our best numerical values for Ω_c, α_F, α_{min}, α_{top}, α_{max} and q from both the scaling relation (3) and from the sensitivity function (7) for various z. It is worthy to mention that we numerically verify that, for $z \geq 3$, $\alpha_{top} \geq [\alpha_{min}\, \alpha_{max}]^{1/2}$ (the equality appears to hold for $z = 3$).

z	Ω_c	α_F	α_{min}	α_{top}	α_{max}	q (Eq. (3))	q (Eq. (7))
3.0	0.606661063469...	1.289	0.632	1.096	1.895	0.05 ± 0.01	0.05 ± 0.01
3.5	0.629593799039...	1.258	0.599	1.124	2.097	0.16 ± 0.01	0.15 ± 0.01
4.0	0.648669091983...	1.234	0.572	1.167	2.289	0.24 ± 0.01	0.24 ± 0.01
4.5	0.664861001064...	1.218	0.542	1.213	2.440	0.30 ± 0.01	0.30 ± 0.01
5.0	0.678831756505...	1.205	0.516	1.266	2.581	0.36 ± 0.01	0.36 ± 0.01
5.5	0.691048981515...	1.195	0.491	1.314	2.701	0.40 ± 0.01	0.40 ± 0.01
6.0	0.701853340894...	1.185	0.473	1.351	2.838	0.43 ± 0.01	0.44 ± 0.01
7.0	0.720182442561...	1.170	0.438	1.451	3.065	0.49 ± 0.01	0.50 ± 0.01
8.0	0.735233625356...	1.158	0.410	1.518	3.280	0.53 ± 0.01	0.53 ± 0.01



(a)

Fig. 3. The plot of $\ln \xi(N)$ *versus* $\ln N$. (a) for $z = 4.5$ and (b) for $z = 6$.

$1/(1-q)$. In Figure 3, the $z = 4.5$ and $z = 6$ cases have been illustrated; see also Table 1.

In order to check the accuracy of the scaling relation (3), one needs to determine the α_{min} and α_{max} values of the $f(\alpha)$ curve. Therefore, one has to study the structure of the trajectory $\theta_1, \theta_2, ..., \theta_i, ...$ and to estimate the singularity spectrum (strength of singularities α and their fractal dimensions f) of this Cantor-like set. To perform the numerical calculation we truncate the series θ_i at a chosen Fibonacci number F_n (we recall that F_n/F_{n+1} gives the golden mean and therefore defines the proper scaling factor). The distances l_i between consecutive points of the set define the natural scales for the partition with measures $p_i = 1/F_n$ attributed to each segment. After that, the singularity spectrum can be directly obtained following a standard prescription [15]. In general, $\sup_\alpha f(\alpha) = d_f$ and, in the present case, $f(\alpha_{min}) = f(\alpha_{max}) = 0$. We define α_{top} through $f(\alpha_{top}) = d_f$.

However, the situation for the generalized circle map is somewhat different than that of the standard circle map in the sense that the standard one has a fast convergence of the $f(\alpha)$ curve when larger number of iterations are considered, whereas the generalized map presents only a slow and oscillatory convergence. An example of a sequence of $f(\alpha)$ curves obtained from increasing number of iterations is shown in Figure 4. Notice its non-monotonic behavior, specially near the upper edge. In Figure 5, we plot the numerically obtained values of α_{min}, α_{top} and α_{max} as a function of $1/\ln N$, where N is the number of iterations. From these data, we are not able to accurately estimate their asymptotic values for large map inflexion z. In Figure 6 we plot the extrapolated $f(\alpha)$ curves for typical

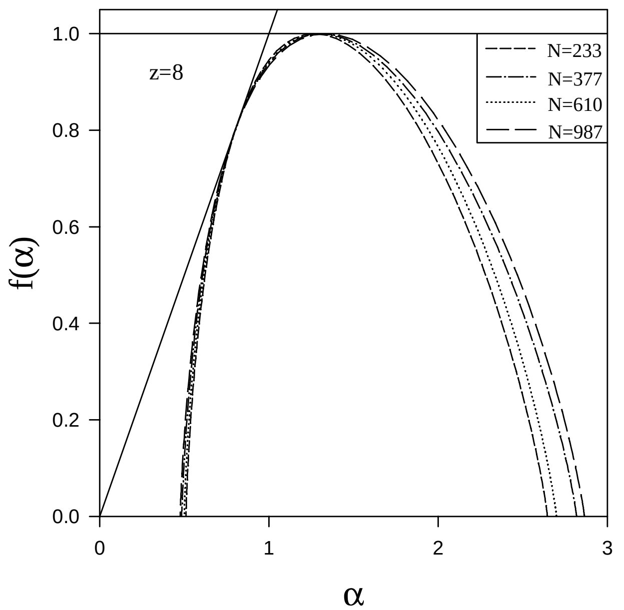


Fig. 4. Approximate multifractal singularity spectra of the critical attractor of the $z = 8$ generalized circular map for increasingly large number of iterations, going from $N = 233$ to 987.

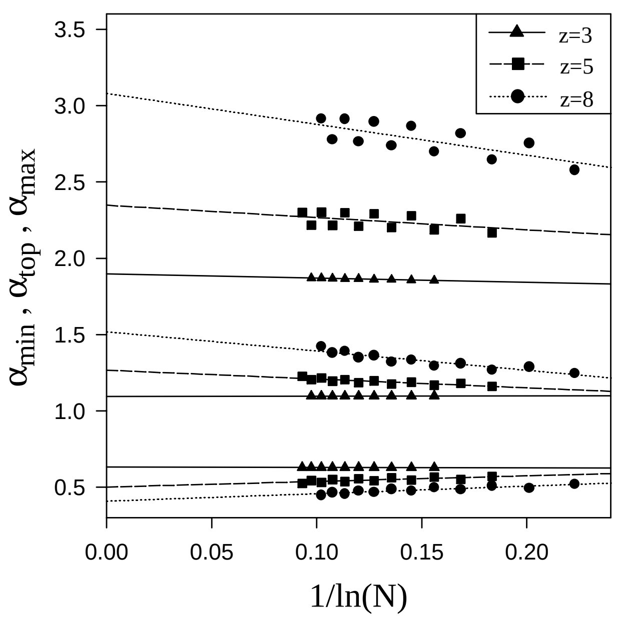


Fig. 5. $\alpha_{\rm min}$ (bottom three lines), $\alpha_{\rm top}$ (middle three lines) and $\alpha_{\rm max}$ (top three lines) from the singularity spectra obtained from distinct number of iterations and typical values of z. Notice the slow and oscillatory convergence.

values of z. Although these show the main expected trends of the singularity spectra, namely z-dependent shape but $d_f = 1$ for all z, their extremal points may need further corrections. We shall point out that this feature is inherent to the numerical method used to estimate the $f(\alpha)$ curve representing the singularity strengths of a multifractal measure. Its extremal points are governed by the scaling behavior of the most concentrated ($\alpha_{\rm min}$) and most

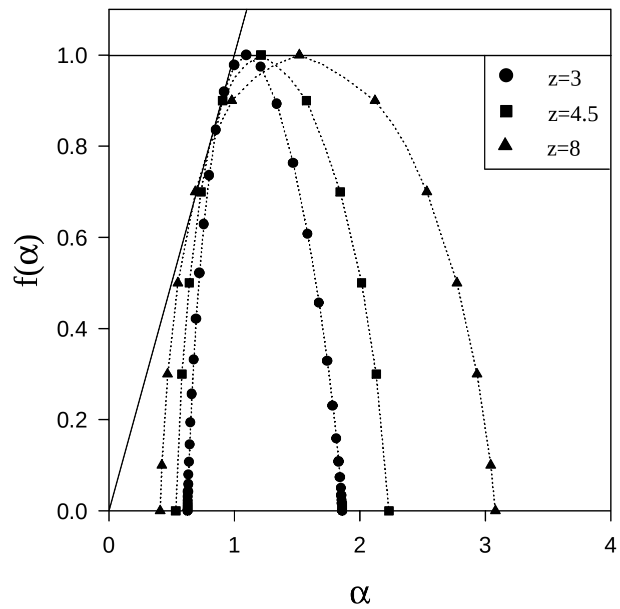


Fig. 6. The extrapolated $f(\alpha)$ curves for typical values of the map inflexion z. Notice that, although the shape is z-dependent, they present $d_f = 1$ for all z. The solid lines correspond to $f(\alpha) = 1$ and $f(\alpha) = \alpha$. The dotted lines are guides to the eye.

rarefied ($\alpha_{\rm max}$) sets in the measure, the latter being usually poorly sampled. Further, this oscillatory convergence may also be related to the non-analyticity of the present generalized circle map which has inflexion z only at the right side of $\theta = 0$, but remains with a cubic inflexion on its left.

An alternative method for computing the extremal values of the singularity strengths α can be obtained by studying how the distances around $\theta = 0$ scale down as the trajectory θ_i is truncated at two consecutive Fibonacci numbers, F_n, F_{n+1}. Shenker has found that this distance shall scale by a universal factor $\alpha_{\rm F}(z)$ [17] (F stands for Feigenbaum). For $z > 3$, the region around the right side of $\theta = 0$ corresponds to the most rarefied one so that $l_{-\infty} \sim [\alpha_F(z)]^{-n}$. The corresponding measure scales as $p_{-\infty} = p_i = 1/F_n \sim (\omega_{\rm GM})^n$, which leads to [8,15,21]

$$\alpha_{\rm max} = \frac{\ln p_{-\infty}}{\ln l_{-\infty}} = \frac{\ln \omega_{\rm GM}}{\ln [\alpha_{\rm F}(z)]^{-1}}. \quad (8)$$

Following along the same lines, the most concentrated region on the set shall scale down as $l_{+\infty} \sim \alpha_{\rm F}(z)^{-zn}$ while $p_{+\infty} \sim [\omega_{\rm GM}]^n$, so that

$$\alpha_{\rm min} = \frac{\ln p_{+\infty}}{\ln l_{+\infty}} = \frac{\ln \omega_{\rm GM}}{\ln [\alpha_{\rm F}(z)]^{-z}}. \quad (9)$$

Equations (8, 9) imply

$$\alpha_{\rm max}/\alpha_{\rm min} = z. \quad (10)$$

In Figure 7 we show, for a large value of the map inflexion ($z = 8$), the values of θ_n, for $\theta_0 = 0$, together with the power-law fitting of the critical sequence of the distances

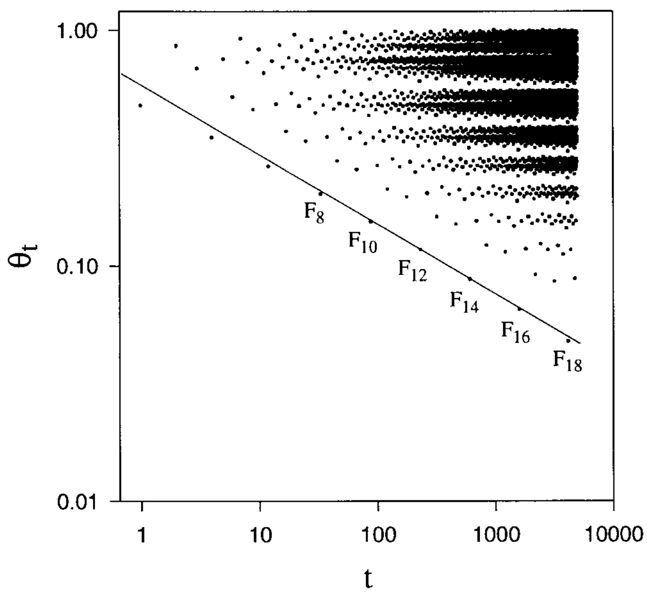


Fig. 7. The sequence θ_t as a function of t for $z = 8$. The minimal distance to $\theta = 0$ scales down as $t^{-0.305}$. Using that $F_n \sim \omega_{\rm GM}^{-n}$ for large n one obtains that $l_{-\infty} \sim \alpha_{\rm F}^{-n}$, with $\alpha_{\rm F}(z = 8) = 1.158$.

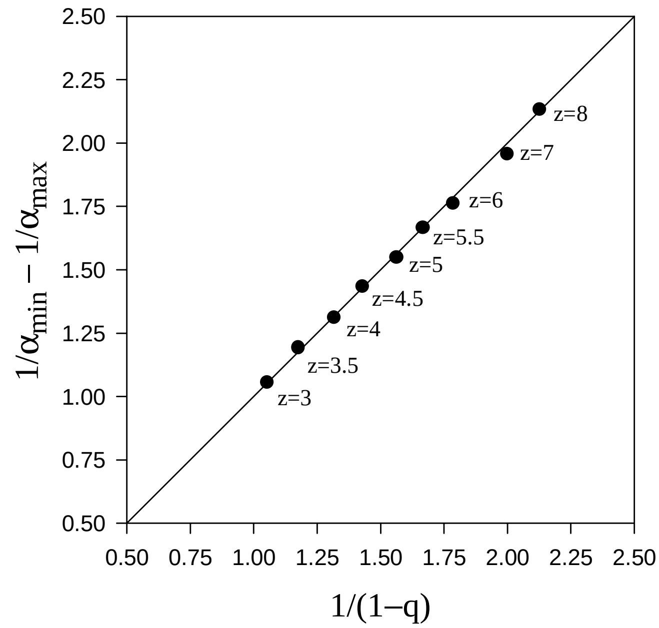


Fig. 8. $1/\alpha_{\min} - 1/\alpha_{\max}$ *versus* $1/(1-q)$ values from the sensitivity function. The straight line represents the scaling prediction (Eq. (3)).

from the right side of $\theta = 0$. The data provide an accurate estimation of the universal factor $\alpha_{\rm F}(z = 8) = 1.1568$ from which precise values of $\alpha_{\min}$ and $\alpha_{\max}$ can be inferred. In Table 1 we list the results for $3 < z < 8$, together with the values of $\alpha_{\rm top}$ from the extrapolated $f(\alpha)$ curves and q obtained from both the scaling relation (3) and from the sensitivity function (7). Notice that the scaling relation (3) is satisfied for all z (see also Fig. 8), just like the case of the logistic-like maps [8]. Furthermore, these results indicate two other important points: (i) Even though the fractal dimension of the support d_f is Euclidean (*i.e.*, $d_f = 1$) for all z, the system sensitivity to initial conditions is still z-dependent. (ii) What matters is $\alpha_{\min}$ and $\alpha_{\max}$, and not d_f, in other words, what precisely controls the entropic index q is not d_f but the sensitivity to initial conditions, reflected by the multifractal nature of the attractor.

4 Conclusions

In this paper, we contribute to the field of low-dimensional dissipative systems by introducing a convenient generalization (with inflexion power z) of the standard circular map and then studying its critical point (analogous to the chaos threshold of logistic-like maps). More precisely, we have numerically studied its sensitivity to the initial conditions and have shown that it is given by a *power*-law ($\xi \propto t^{1/(1-q)}$) instead of the usual *exponential* behavior. Moreover, we have shown that this example, as the logistic-like maps, satisfies the scaling law given in equation (3). Although q, $\alpha_{\min}$ and $\alpha_{\max}$ depend on z, d_f *does not.* This is a quite important result because it illustrates that, for having a Boltzmann-Gibbs scenario ($q = 1$), it is not enough to fully occupy the phase space during the dynamical evolution of the system. What is essentially necessary is to have a *quick*, exponential-like occupation of the phase space, so that ergodicity and mixing are naturally attained.

In addition to this, it is worth mentioning that there are also other efforts along this line which address high-dimensional dissipative systems, namely those exhibiting self-organized criticality [22]. Amongst them, the study of the Bak-Sneppen model for biological evolution [10] and the Suzuki-Kaneko model for the battle of birds defending their territories [11] can be enumerated. In both cases it is shown that, at the self-organized critical state, a power-law sensitivity to initial conditions emerges, like in the present case.

It should be emphasized that these ideas seem to be valid and applicable not only to low- and high-dimensional dissipative systems but also to conservative (Hamiltonian) systems with long-range interactions. This fact has been illustrated very recently [12] on the long-range classical XY ferromagnetic model, whose entire Lyapunov spectrum collapses (for an infinitely wide energy interval) to zero at the thermodynamic limit if (and only if) the range of the interactions is sufficiently long. As a final remark it should be of great interest to extend the present ideas to Hamiltonian systems in the vicinity of a second order critical point which exhibit also a power-law sensitivity to initial conditions without an evident breakdown of Boltzmann-Gibbs scenario. Further efforts focusing, along these lines, both dissipative and conservative systems are welcome.

One of us (UT) is a TUBITAK Münir Birsel Foundation Fellow and acknowledges its financial support which made his visit to CBPF (Rio de Janeiro) possible. He would also like to thank CBPF for kind hospitality during his stay. This work was partially supported by PRONEX and CNPq (Brazilian Agencies) as well as Ege University Research Fund under the project number 98FEN025.

References

1. M. Arik, D.D. Coon, J. Math. Phys. **17**, 524 (1975); L.C. Biedenharn, J. Phys. A **22**, L873 (1989); A.J. MacFarlane, J. Phys. A **22**, 4581 (1989).
2. C. Tsallis, J. Stat. Phys. **52**, 479 (1988); E.M.F. Curado, C. Tsallis, J. Phys. A **24**, L69 (1991); corrigenda: **24**, 3187 (1991); **25**, 1019 (1992). An updated bibliography is accessible at `http://tsallis.cat.cbpf.br/biblio.htm`
3. C. Tsallis, Phys. Lett. A **195**, 329 (1994); S. Abe, Phys. Lett. A **224**, 326 (1997); A. Erzan, Phys. Lett. A **225**, 235 (1997) ; S.F. Özeren, U. Tırnaklı, F. Büyükkılıç, D. Demirhan, Eur. Phys. J. B **2**, 101 (1998).
4. A.M. Mariz, Phys. Lett. A **165**, 409 (1992); J.D. Ramshaw, Phys. Lett. A **175**, 169 and 171 (1993); A. Plastino, A.R. Plastino, Phys. Lett. A **177**, 177 (1993); A.R. Plastino, A. Plastino, Physica A **202**, 438 (1994); J.D. Ramshaw, Phys. Lett. A **198**, 119 (1995); C. Tsallis, Phys. Lett. A **206**, 389 (1995); E.P. da Silva, C. Tsallis, E.M.F. Curado, Physica A **199**, 137 (1993); **203**, E 160 (1994); A. Chame, E.M.L. de Mello, J. Phys. A **27**, 3663 (1994); D.A. Stariolo, Phys. Lett. A **185**, 262 (1994); A. Plastino, C. Tsallis, J. Phys. A **26**, L893 (1993); A.K. Rajagopal, Phys. Rev. Lett. **76**, 3469 (1996); A. Chame, E.V.L. de Mello, Phys. Lett. A **228**, 159 (1997).
5. A.R. Plastino, A. Plastino, Phys. Lett. A **174**, 384 (1993); **193**, 251 (1994); B.M. Boghosian, Phys. Rev. E **53**, 4754 (1996); P.A. Alemany, D.H. Zanette, Phys. Rev. E **49**, R956 (1994); C. Tsallis, S.V.F. Levy, A.M.C. de Souza, R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995); Erratum: Phys. Rev. Lett. **77**, 5442 (1996); D.H. Zanette, P.A. Alemany, Phys. Rev. Lett. **75**, 366 (1995); M.O. Caceres, C.E. Budde, Phys. Rev. Lett. **77**, 2589 (1996); D.H. Zanette, P.A. Alemany, Phys. Rev. Lett. **77**, 2590 (1996); L.S. Lucena, L.R. da Silva, C. Tsallis, Phys. Rev. E **51**, 6247 (1995); C. Anteneodo, C. Tsallis, J. Mol. Liq. **71**, 255 (1997); A. Lavagno, G. Kaniadakis, M. Rego-Monteiro, P. Quarati, C. Tsallis, Astrophys. Lett. Comm. **35**, 449 (1997); C. Tsallis, F.C. Sa Barreto, E.D. Loh, Phys. Rev. E **52**, 1447 (1995); U. Tırnaklı, F. Büyükkılıç, D. Demirhan, Physica A **240**, 657 (1997); A.R. Plastino, A. Plastino, H. Vucetich, Phys. Lett. A **207**, 42 (1995); U. Tırnaklı, F. Büyükkılıç, D. Demirhan, Phys. Lett. A **245**, 62 (1998); D.F. Torres, H. Vucetich, A. Plastino, Phys. Rev. Lett. **79**, 1588 (1997); Erratum: **80**, 3889 (1998).
6. C. Tsallis, A.R. Plastino, W.-M. Zheng, Chaos, Solitons and Fractals **8**, 885 (1997).
7. U.M.S. Costa, M.L. Lyra, A.R. Plastino, C. Tsallis, Phys. Rev. E **56**, 245 (1997).
8. M.L. Lyra, C. Tsallis, Phys. Rev. Lett. **80**, 53 (1998).
9. M.L. Lyra, *Weak chaos: Power-law sensitivity to initial conditions and nonextensive thermostatistics*, Ann. Rev. Comp. Phys., edited by D. Stauffer (World Scientific, Singapore, 1999), Vol. 6, p. 31.
10. F.A. Tamarit, S.A. Cannas, C. Tsallis, Eur. Phys. J. B **1**, 545 (1998).
11. J. Suzuki, K. Kaneko, Physica D **75**, 328 (1994); A.R.R. Papa, C. Tsallis, Phys. Rev. E **57**, 3923 (1998).
12. C. Anteneodo, C. Tsallis, Phys. Rev. Lett. **80**, 5313 (1998).
13. P. Grassberger, M. Scheunert, J. Stat. Phys. Phys. **26**, 697 (1981).
14. R.C. Hilborn, *Chaos and Nonlinear Dynamics* (Oxford University Press, New York, 1994), p. 390.
15. T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, B.I. Shraiman, Phys. Rev. A **33**, 1141 (1986).
16. M.H. Jensen, P. Bak, T. Bohr, Phys. Rev. Lett. **50**, 1637 (1983); Phys. Rev. A **30**, 1960 (1984); **30**, 1970 (1984).
17. S.J. Shenker, Physica D **5**, 405 (1982).
18. M.J. Feigenbaum, L.P. Kadanoff, S.J. Shenker, Physica D **5**, 370 (1982).
19. M.H. Jensen, P. Bak, T. Bohr, Phys. Rev. A **30**, 1960 (1984).
20. R. Delbourgo, B.G. Kenny, Phys. Rev. A **42**, 6230 (1990).
21. M.J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979).
22. P. Bak, C. Tang, K. Weisenfeld, Phys. Rev. Lett. **59**, 381 (1987).